REMARKS ON ISOVARIANT MAPS FOR REPRESENTATIONS (Topological Transformation Groups and Related Topics)

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1. Introduction

In this note we shall discuss an isovariant version of the Borsuk-Ulam theorem, which we call the isovariant Borsuk-Ulam theorem, and give some related results on the isovariant Borsuk-Ulam theorem for $SO(3)$.

We say that a compact Lie group G has the *IB-property* if G has the following property:

- For any (orthogonal) G-representations V, W such that a G-isovariant map $f: V \to W$ exists, the inequality
$$\dim V - \dim V^G \leq \dim W - \dim W^G$$
holds.

An interesting problem is the following.

Problem A. Which compact Lie groups have the IB-property?

By a result of Wasserman [3], any compact solvable Lie group has the IB-property, however this problem is still open for a general compact Lie group. On the other hand, a weaker version of this problem has an affirmative answer for an arbitrary compact Lie group.

Theorem 1.1 (The weak isovariant Borsuk-Ulam theorem). *For an arbitrary compact Lie group, the weak isovariant Borsuk-Ulam theorem holds.*

In section 2 we shall recall this theorem from [2].

In section 3, as an example, we shall discuss further details when $G = SO(3)$, and show the isovariant Borsuk-Ulam theorem holds when the dimension of $SO(3)$-representation is small, that is,

Proposition 1.2. *Let $V = \oplus_{i=0}^{6} a_i U_i \oplus U$ and $W = \oplus_{i=0}^{6} b_i U_i \oplus U$, where a_i, b_i are nonnegative integers, U_i is the $(2i+1)$-dimensional irreducible $SO(3)$-representation and U is any $SO(3)$-representation. If there is an $SO(3)$-isovariant map from V to W, then*
$$\dim V - \dim V^{SO(3)} \leq \dim W - \dim W^{SO(3)}$$

2. A weak version of the isovariant Borsuk-Ulam theorem

We first recall the *prime condition* in order to state Wasserman's result.

Definition 1. We say that a finite group G satisfies the *prime condition* if for every pair of subgroups $H \lhd K$ with K/H simple,

$$\sum_{\substack{p:\text{prime} \\ p \mid |g|}} \frac{1}{p} \leq 1$$

for every $g \in K/H$, where $|g|$ denotes the order of g.

Wasserman's isovariant Borsuk-Ulam theorem is stated as follows.

Theorem 2.1 (The isovariant Borsuk-Ulam theorem). *Every finite group G satisfying the prime condition has the IB-property.*

Remark. All finite groups do not satisfy the prime condition, for example, A_n, $n \leq 11$, satisfies the prime condition, but A_n, $n \geq 12$, does not satisfy the prime condition. The author does not know whether all A_n have the IB-property.

We next consider a weaker version of the isovariant Borsuk-Ulam theorem.

Definition 2. We say that a compact Lie group G has the *WIB-property* if there exists a monotone increasing function $\varphi_G : \mathbb{N}_0 \to \mathbb{N}_0$ ($\mathbb{N}_0$: the nonnegative integers) diverging to $+\infty$ with the following property:

- For any (orthogonal) G-representations V, W such that a G-isovariant map $f : V \to W$ exists, the inequality
$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G$$
holds.

Remark. In [2] we defined the WIB-property for linear G-spheres, but it is essentially same as above, because one can see that the existence of a G-isovariant map from V to W and the existence of a G-isovariant map from SV to SW are equivalent.

A weak version of Problem A is:

Problem B. Which compact Lie groups have the WIB-property?

The answer is the following:

Theorem 2.2 (The weak isovariant Borsuk-Ulam theorem). *An arbitrary compact Lie group G has the WIB-property.*

The outline of proof is as follows. The full details will appear in [2]. We first note:

Lemma 2.3. *Let*

$$1 \to H \to G \to K \to 1$$

be a short exact sequence of compact Lie groups.

(1) *If H and K have the WIB [IB]-property, then G has the WIB [IB]-property.*

(2) *If G has the WIB [IB]-property, then K has the WIB [IB]-property.*

By this lemma, the problem is reduced to two cases:

(1) G is a finite simple group,
(2) G is a compact, simply-connected, simple Lie group.

Using the (ordinary) Borsuk-Ulam theorem, one can see

Proposition 2.4. *C_p (p : prime) and S^1 have the IB-property.*

Therefore we obtain the following corollary from Lemma 2.3 and Proposition 2.4:

Corollary 2.5. *Any compact solvable Lie group has the IB-property.*

The next result is easy, but plays an important role in the proof of the weak isovariant Borsuk-Ulam theorem.

Lemma 2.6. *Let H be a closed subgroup of G with the IB-property. Assume that there exists a constant $0 < c < 1$ such that $\dim U^H \leq c \dim U$ for all nontrivial irreducible representations U of G. Then G has the WIB-property, and moreover $\varphi_G(n)$ can be taken to be $\langle (1-c)n \rangle$, where $\langle x \rangle = \min\{n \in \mathbb{Z} \mid n \geq x\}$.*

Proof. Let $f : V \to W$ be any G-isovariant map between representations. Let $V = V_G \oplus V^G$ and $W = W_G \oplus W^G$, where V_G [resp. W_G] denotes the orthogonal complement of V^G [resp. W^G]. Since the natural inclusion $i : V_G \to V$ and the projection $p : W \to W_G$ are G-isovariant, we get a G-isovariant map $g := p \circ \tilde{f} \circ i : V_G \to W_G$. Since H has the IB-property, it follows that

$$\dim V_G - \dim V_G^H \leq \dim W_G - \dim W_G^H \leq \dim W_G.$$

By the complete reducibility of G, V_G is isomorphic to a direct sum of nontrivial irreducible representations. Hence by assumption one can see that

$$(1-c) \dim V_G \leq \dim V_G - \dim V_G^H.$$

Setting $\varphi_G(n) = \langle (1-c)n \rangle$, we obtain that $\varphi_G(\dim V_G) \leq \dim W_G$, or equivalently

$$\varphi_G(\dim V - \dim V^G) \leq \dim W - \dim W^G.$$

Clearly φ_G is a monotone increasing function diverging to ∞. This implies that G has the WIB-property.

In the case (1), since there are only finitely many irreducible representations, we have following:

Proposition 2.7. *Let G be a finite simple group. Let H be any nontrivial subgroup of G. Then there exists a constant $0 < c < 1$ such that $\dim U^H \leq c \dim U$ for all nontrivial irreducible representations U.*

In particular, taking H as a cyclic subgroup of prime order, we obtain by Lemma 2.6 that G has the WIB-property.

In the case (2), by representation theory of compact Lie groups, we also see the following:

Proposition 2.8 ([2]). *Let G be a compact, simply-connected, simple Lie group and T a maximal torus. There exists a constant $0 < c < 1$ such that $\dim U^T \leq c \dim U$ for all nontrivial irreducible representations U of G.*

Since T has the IB-property, it follows from Lemma 2.6 that G has the WIB-property. Thus the proof of the weak isovariant Borsuk-Ulam theorem is complete.

Before ending this section, we give a remark on the (weak) isovariant Borsuk-Ulam theorem in semilinear actions.

Definition 3. A closed (smooth) G-manifold M is called a *semilinear G-sphere* if the H-fixed point set M^H is homotopy equivalent to a sphere or empty for every closed subgroup H of G.

We can consider a similar problem in the family of semilinear G-spheres, however the conclusion is different from linear case. For semilinear G-spheres, the (weak) isovariant Borsuk-Ulam theorem does not hold in general. In this case we show in [2] that the (weak) isovariant Borsuk-Ulam theorem holds if and only if G is solvable.

3. Some estimate of φ_G for $G = SO(3)$

In this section we concerned with the function φ_G as in Definition 2.

We set

$$c_G(n) = \max\{\varphi_G(n) \mid \varphi_G \text{ as in Definition 2}\}$$

for $n \geq 1$, and $c_G(0) = 0$ for convenience.

Set $\mathcal{D}_G = \{n \mid n = \dim V - \dim V^G \text{ for some } V\}$. We also define a similar function d_G on $\mathcal{D}_G$, where $d_G(n)$, $n \geq 1$, is defined as the greatest integer with the following property:

- For any representation V with $\dim V - \dim V^G = n$ and for any W, if there is a G-isovariant map $f : V \to W$, then
$$d_G(n) \leq \dim W - \dim W^G$$
holds.

We also define $d_G(0) = 0$. Though the definition of d_G resembles that of c_G, these are different in definition, namely d_G need not be monotonely increasing. (However the author does not have such an example.)

We first note the following.

Lemma 3.1. *The value $c_G(n)$, $n \geq 1$, is equal to the greatest integer with the following property:*

- *For any representation V with $\dim V - \dim V^G \geq n$ and for any W, if there is a G-isovariant map $f : V \to W$, then*
$$c_G(n) \leq \dim W - \dim W^G$$
holds.

Proof. Let $c'_G(n)$ be the greatest integer satisfying the above property. Then c'_G is monotonely increasing and diverging to ∞ by the weak isovariant Borsuk-Ulam theorem. Hence c'_G is one of φ_G and so $c'_G = c_G$.

Remark. From this lemma, c_G is thought of as an isovariant version of the Borsuk-Ulam function b_G defined in [1].

One can easily see the following by definition.

Proposition 3.2. *$\varphi_G(n) \le c_G(n) \le d_G(n) \le n$ for any $n \in \mathcal{D}_G$.*

Proposition 3.3. *The following are equivalent.*

(1) *G has the IB-property.*
(2) *$c_G(n) = n$ for any $n \in \mathcal{D}_G$.*
(3) *$d_G(n) = n$ for any $n \in \mathcal{D}_G$.*

As an example we shall estimate c_G or d_G by finding some function φ_G when $G = SO(3)$. As is well-known, $SO(3)$ has only one (real) $(2k+1)$-dimensional irreducible representation for each $k \ge 0$, which we denote by U_k. Let T ($\cong S^1$) be a maximal torus and N ($\cong O(2)$) the normalizer of T. Each U_k has the weight $1 + t + \cdots + t^k$, where t is the standard irreducible representation of S^1. So we obtain $\dim U_k^T = 1$, moreover we have

$$\dim U_k^N = \begin{cases} 1 & (k : \text{even}) \\ 0 & (k : \text{odd}), \end{cases}$$

and so

$$\frac{\dim U_k^N}{\dim U_k} = \begin{cases} \frac{1}{2k+1} & (k : \text{even}) \\ 0 & (k : \text{odd}). \end{cases}$$

Therefore we obtain

$$\dim V^N \le \frac{1}{5} \dim V$$

for any representation V with $V^G = 0$. Since N is solvable, by Proposition 2.8 and its proof, we obtain

$$\frac{4}{5}(\dim V - \dim V^G) \le \dim W - \dim W^G.$$

So φ_G can be taken as

$$\varphi_G(n) = \left\langle \frac{4}{5} n \right\rangle.$$

and hence

$$c_G(n) \ge \left\langle \frac{4}{5} n \right\rangle.$$

For $G = SO(3)$, $\mathcal{D}_G$ consists of the nonnegative integers except $n = 1, 2, 4$. Consequently we have $c_G(3) = 3$, $c_G(5) \ge 4$, $c_G(6) \ge 5$, etc. However this estimate is not very sharp. In fact one can see $c_G(5) = 5$, $c_G(6) = 6$ later.

Remark. The value of φ_G or c_G of $n \notin \mathcal{D}_G$ is not important as well as of $n = 0$ for our purpose.

The following is a partial result on the isovariant Borsuk-Ulam theorem for

Proposition 3.4. *Let $G = SO(3)$. Let $V = \oplus_{i=0}^{6} a_i U_i \oplus U$ and $W = \oplus_{i=0}^{6} b_i U_i \oplus U$, where a_i, b_i are nonnegative integers and U is any representation. If there is a G-isovariant map from V to W, then*

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

We notice some facts for the sake of proof. Firstly it suffices to show the proposition when $a_0 = b_0 = 0$. Secondly, as is well-known, the (closed) proper subgroups of $SO(3)$ are the following: the cyclic group C_n, the dihedral group D_n, the tetrahedral group T, the octahedral group O, the icosahedral group I, $SO(2)$ and $O(2)$. All of these except I are solvable, and I is isomorphic to A_5, whence all proper subgroups of $SO(3)$ have the IB-property. Therefore the isovariant Borsuk-Ulam theorem gives various inequalities between dimensions. We consider them in a general setting. Let $V = \oplus_{i=1}^{n} a_i U_i$ and $W = \oplus_{i=1}^{n} b_i U_i$. Set $\eta = W - V$ and set $\alpha_i = \sum_{k=i}^{n}(b_k - a_k)$, $1 \leq i \leq n$. Then we have

$$\mathrm{Res}_{SO(2)}\eta = \alpha_1 1 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n,$$

and

$$\dim \eta = 3\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n).$$

By the isovariant Borsuk-Ulam theorem, one can easily see the following.

Lemma 3.5. (1) $\dim \eta^{SO(2)} - \dim \eta^{O(2)} = \sum_{k=1}^{n}(-1)^{k-1}\alpha_k \geq 0$.

(2) $\dim \eta - \dim \eta^{C_p} = \sum_{k \not\equiv 0(p)} \alpha_k \geq 0$.

(3) $\dim \eta^{C^2} - \dim \eta^{C^4} = \sum_{\substack{k \equiv 0(2) \\ k \not\equiv 0(4)}} \alpha_k \geq 0$.

(4) *If $i > \frac{n}{3}$, then $\alpha_i \geq 0$.*

Proof. (1)–(3): easy.

(4): By the isovariant Borsuk-Ulam theorem, we have

$$\dim \eta^{C_i} - \dim \eta^{C_{2i}} = 2(\alpha_i + \alpha_{3i} + \alpha_{5i} + \cdots) \geq 0.$$

Since $3i > n$, α_m must be 0 for $m \geq 3i$. Hence $\alpha_i \geq 0$.

Proof of Proposition 3.4. We may suppose that $a_0 = b_0 = 0$. When $n = 6$, by Lemma 3.5, we have inequalities

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6 \geq 0,$$
$$\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 \geq 0,$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 \geq 0,$$
$$\alpha_2 + \alpha_6 \geq 0.$$

Adding up these inequalities, we have

$$3\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 \geq 0.$$

Since $\alpha_4 \geq 0$ and $\alpha_6 \geq 0$ by Lemma 3.5 (4), it follows that

$$\dim \eta = 3\alpha_1 + 2(\alpha_2 + \cdots + \alpha_6) \geq 0.$$

Hence $\dim V \leq \dim W$.

Remark. For a general n, it does not seem that the above argument works well though many other inequalities as in Lemma 3.5 exist.

Proposition 3.4 gives some information about $c_{SO(3)}(n)$ or $d_{SO(3)}(n)$ for lower n. For example,

Example 3.6. $d_{SO(3)}(n) = n$ for $n \leq 15$ $(n \in \mathcal{D}_{SO(3)})$.

Proof. When $n \leq 14$, $d_{SO(3)}(n) = n$ follows directly from Proposition 3.4. If $d_{SO(3)}(15) < 15$, there is a G-isovariant G-map $f : S(V) \to S(W)$ for some V, W $(V^G = W^G = 0)$ such that $\dim W < \dim V = 15$, hence W does not include U_k, $k > 6$, by dimensional reason. Since $\alpha_7 = b_7 - a_7 \geq 0$ by Lemma 3.5 (4), V does not also include U_7. Hence $d_{SO(3)}(15) = 15$ by Proposition 3.4.

By a similar argument we also have

Example 3.7. $c_{SO(3)}(n) = n$ for $n \leq 15$ $(n \in \mathcal{D}_{SO(3)})$.

Remark. By a further argument, one can see that the above equality holds for some more large integers. The detail is left to the readers.

Finally we pose

Conjecture. *$c_G(n) = d_G(n) = n$ for each $n \in \mathcal{D}_G$ when $G = SO(3)$.*

References

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